Setup: FI : category of finite sets and injections

k : commutative ring

FI-module over k : A functor $V : \mathrm{FI} \longrightarrow k\text{-Mod}$

$$S \longmapsto V_S$$

$$V_n := V_{\{1,\dots,n\}}$$

$$\begin{tikzcd}
V_0 \arrow[r] & V_1 \arrow[r, shift left] \arrow[r, shift right] & V_2 \arrow[loop below, "\mathfrak{S}_2"] \arrow[r, shift left=5] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, shift right=5] & V_3 \arrow[loop below, "\mathfrak{S}_3"] \arrow[r, shift left=4] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=4] & \cdots & V_n \arrow[loop below, "\mathfrak{S}_n"] \arrow[r, shift left=4] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=4] & {}
\end{tikzcd}$$

$\mathfrak{S}_n$: symmetric group on n letters.

Motivating example: Given a topological space X, there is a functor

$$\begin{aligned} PConf_\bullet(X) : FI^{op} &\longrightarrow Top \\ S &\longmapsto Emb(S, X) \end{aligned}$$ "co-FI-space"

$$\left(n \longmapsto \{(x_1, \dots, x_n) : x_i \in X,\ x_i \neq x_j\} \right)$$

Hence given $j \in \mathbb{N}$,

$$H^j(PConf_\bullet(X); k) : FI \longrightarrow k\text{-}Mod.$$

Church – Ellenberg – Farb 2015:

- If k is a field of characteristic zero and
$$V : FI \longrightarrow k\text{-Mod}$$
is finitely generated, then the decomposition of V_n's into (irreducible) Specht modules satisfy "Specht stability" (last talk, defined via adding a box to the 1st row)

- For a conn. oriented d-manifold with $d \geq 2$, $\dim_k H^*(M;k) < \infty$ and $j \in \mathbb{N}$, the FI-module $H^j(PConf_\bullet(M);k)$ is finitely generated.

FB: the category of finite sets and bijections. $\subseteq$ FI

An FB-module is simply a sequence $n \mapsto V_n$ of $\mathfrak{S}_n$-rep's, no transition maps.

$\mathrm{Res}^{FI}_{FB} : [FI, k\text{-Mod}] \longrightarrow [FB, k\text{-Mod}]$ has a left adjoint

$\mathrm{Ind}^{FI}_{FB} : [FB, k\text{-Mod}] \longrightarrow [FI, k\text{-Mod}]$.

An FI-module V is called <u>induced</u> if $V \cong \mathrm{Ind}^{FI}_{FB}(W)$ for some FB-module W.

Given a morphism $f: S \hookrightarrow T$ in FI and $W: FB \to k\text{-Mod}$, we have

$$\begin{tikzcd}
\mathrm{Ind}_{FB}^{FI}(W)_S \arrow[d] \arrow[r, phantom, "="] & \bigoplus_{A \subseteq S} W_A \arrow[d, "\text{sum of maps } W_A \to W_{f(A)}"] \\
\mathrm{Ind}_{FB}^{FI}(W)_T \arrow[r, phantom, "="] & \bigoplus_{B \subseteq T} W_B
\end{tikzcd}$$

Fact : $\mathrm{Ind}_{FB}^{FI}(W)$ is fin. gen. $\Leftrightarrow$ W is fin. gen. $\Leftrightarrow$ $\forall n$ W_n is fin. gen. $W_n = 0$ for $n >> 0$.

If W satisfies $W_A = 0$ for $|A| \neq r$,

$$\mathrm{Ind}_{FB}^{FI}(W)_n = \bigoplus_{\substack{A \subseteq \{1,\dots,n\} \\ |A| = r}} W_A = \mathrm{Ind}_{\mathfrak{S}_r \times \mathfrak{S}_{n-r}}^{\mathfrak{S}_n} (W_r \boxtimes \mathbb{1}_{n-r})$$

If k is a field of char $=0$, it suffices to understand the case

$$W_r = S_k(\mu) \text{ for } \mu \vdash r$$

(irreducible $k\mathfrak{S}_r$-module indexed by a partition μ of size r)

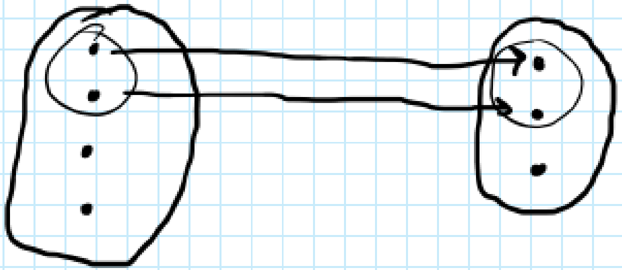
Pieri's Rule: For a field k of char $=0$, a partition $\mu \vdash r$ and $n \in \mathbb{N}$,

$$\mathrm{Ind}_{\mathfrak{S}_r \times \mathfrak{S}_{n-r}}^{\mathfrak{S}_n} \left(S_k(\mu) \boxtimes \mathbb{1}_{n-r} \right) \cong \bigoplus_{\lambda \vdash n} S_k(\lambda)$$

$\lambda = (n-r)$ boxes added to μ in different columns

Hemmer's Observation: Fixing k, μ, r and varying n above, the resulting sequence of $\mathfrak{S}_n$-reps over k above satisfy Specht stability in the range $n \geq r + \mu_1$, sharply.

Let $FI_\#$: category of finite sets and partial bijections. $\supseteq FI$



<u>Note</u>: Given $W : FB \longrightarrow k\text{-Mod}$, the induced FI-module $\mathrm{Ind}_{FB}^{FI}(W)$ naturally extends to an $FI_\#$-module

$$
\begin{tikzcd}
\mathrm{Ind}_{FB}^{FI}(W) : FI \arrow[r] & k\text{-Mod} \\
FI_\# \arrow[u, hook] \arrow[ur, dashed] &
\end{tikzcd}
$$

[CEF 2015]: The functor

$$\mathrm{Ind}_{FB}^{FI} : [FB, k\text{-Mod}] \longrightarrow [FI_\#, k\text{-Mod}]$$

is an equivalence of categories. (for any ring k)

[CEF 2015] + [Miller–Wilson 2017]: If M is a non-compact d-manifold with $d \geq 2$, writing hTop: homotopy category of top. sp.

$$\begin{tikzcd} FI^{op} \arrow[rr, "PConf_\bullet(M)"] \arrow[d, hook] & & Top \arrow[r, two heads] & hTop \\ FI_\#^{op} \arrow[rrr, dashed, "\exists"'] & & & \end{tikzcd}$$

Corollary: For non-compact d-manifold M with $d \geq 2$, $\forall j \in \mathbb{N}$ $\exists\, W^j(M): FB \longrightarrow k\text{-Mod}$ such that

$$H^j(PConf_\bullet(M); k) \cong \mathrm{Ind}_{FB}^{FI}(W^j(M))$$

over any (commutative) ring k.

If M is also connected, $W^j(M)_n = 0$ for $n > \begin{cases} j & \text{if } d \geq 3 \\ 2j & \text{if } d = 2 \end{cases}$.

So $H^j(PConf_n(M); k) \cong \bigoplus_{r=0}^{2j} \mathrm{Ind}_{\mathfrak{S}_r \times \mathfrak{S}_{n-r}}^{\mathfrak{S}_n}(W^j(M)_r \boxtimes \mathbb{1}_{n-r})$