THE FUNDAMENTAL GROUPOID

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I will try to write the very basics of the fundamental groupoid theory myself to understand the material better and internalize it that caters best to my understanding.

Occasionally I will forget writing "continuous" but the maps will be continuous.

1. The Path Category

Given a topological space, there is an immediate way to construct a small category out of it where the objects are points of X and the morphisms are paths between objects (no homotopies yet! merely paths). One needs to be careful here, if we were to define a path to be a continuous map which always has domain $[0,1]$ and define the composition of two paths by "speeding them up" to make them fit in a $[0,1]$ domain, we wouldn't have associativity because the speeding up will be uneven when paths are composed in different ways. So we make the more general definition that a path in a space X is a continuous map

$$\gamma : [0,a] \to X$$

where $a \geq 0$. We call a the *duration* of the path and denote it by $|\gamma|$. We call $\gamma(0)$ the *source* of γ and denote it by $\mathbf{s}(\gamma)$ and $\gamma(a)$ is the *target* of γ, denoted by $\mathbf{t}(\gamma)$.[1] Finally, given $x \in X$ we denote the constant path

$$\begin{aligned} \{0\} = [0,0] &\to X \\ 0 &\mapsto x \end{aligned}$$

by $\mathbf{c}_x$.

Now we can define a well-behaved composition. Given paths γ and δ on a space X with $\mathbf{t}(\gamma) = \mathbf{s}(\delta)$, we define

$$\begin{aligned} \delta \cdot \gamma : [0, |\gamma| + |\delta|] &\to X \\ t &\mapsto \begin{cases} \gamma(t) & \text{if } t \in [0, |\gamma|] \\ \delta(t - |\gamma|) & \text{if } t \in [|\gamma|, |\gamma| + |\delta|] \end{cases} \end{aligned}$$

which is well-defined since $\gamma(|\gamma|) = \mathbf{t}(\gamma) = \mathbf{s}(\delta) = \delta(0)$ and continuous by the pasting lemma. Now it is straightforward to check that $\mathcal{P}(X)$ is a category with the assignments

- $\mathrm{Obj}(\mathcal{P}(X)) = X$,
- $\mathrm{Mor}(\mathcal{P}(X)) =$ all paths in X,
- $\mathbf{s} : \mathrm{Mor}(\mathcal{P}(X)) \to \mathrm{Obj}(\mathcal{P}(X))$ is the domain map,
- $\mathbf{t} : \mathrm{Mor}(\mathcal{P}(X)) \to \mathrm{Obj}(\mathcal{P}(X))$ is the codomain map,
- The map $\mathrm{Obj}(\mathcal{P}(X)) \to \mathrm{Mor}(\mathcal{P}(X))$ given by $x \mapsto \mathbf{c}_x$ is the map assigning every object to the identity morphism on that object,

[1] I am afraid these will bite me in the back when I use s and t as time or position parameters. Well, that's what macros are for!

with the composition as defined above.

Let $f : X \to Y$ be a continuous map. Then if γ is a path in X, then $f \circ \gamma$ is a path in X. Note that $f \circ \mathbf{c}_x = \mathbf{c}_{f(x)}$ and if γ, δ are two paths in X with $\mathbf{s}(\delta) = \mathbf{t}(\gamma)$, then

$$\begin{aligned}(f \circ (\delta \cdot \gamma))(t) &= f\left(\begin{cases}\gamma(t) & \text{if } t \in [0, |\gamma|] \\ \delta(t - |\gamma|) & \text{if } t \in [|\gamma|, |\gamma| + |\delta|]\end{cases}\right) \\ &= \begin{cases}(f \circ \gamma)(t) & \text{if } t \in [0, |f \circ \gamma|] \\ (f \circ \delta)(t - |\gamma|) & \text{if } t \in [|f \circ \gamma|, |f \circ \gamma| + |f \circ \delta|]\end{cases} \\ &= ((f \circ \delta) \cdot (f \circ \gamma))\,(t)\,.\end{aligned}$$

Thus f gives rise to a functor $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$. The assignment $f \mapsto f_*$ is also functorial so we have a functor

$$\mathcal{P} : \mathsf{Top} \to \mathsf{Cat}$$

which assigns each space its path category and sends each continuous map f to f_*.

2. Congruence Relations and Homotopy

Our next aim is to mod-out the path category by path homotopies and the result will be the fundamental groupoid. This modding out is an instance of taking the quotient of a category under a *congruence relation*. Here is the definition: Let C be an arbitrary (not necessarily small, but locally small for sanity's sake) category. The datum of a congruence relation $\sim$ on C is a collection of equivalence relations $\sim_{X,Y}$ on $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ for every pair of objects X, Y which respects composition. That is, given $f_1 \sim f_2$ (so f_1, f_2 have the same domain and codomain) we have

- if $\mathrm{dom}(g) = \mathrm{cod}(f_i)$, then $g \circ f_1 \sim g \circ f_2$.
- if $\mathrm{dom}(f_i) = \mathrm{cod}(h)$, then $f_1 \circ h \sim f_2 \circ h$.

Since $\sim$ is reflexive and transitive, a more compact way of saying this is that if there are two pairs of related morphisms which are composable, then all four possible compositions are related. This observation suffices to see that we can define a well-defined composition to yield a new category $\mathsf{C}/\sim$ with

- $\mathrm{Obj}(\mathsf{C}/\sim) = \mathrm{Obj}(\mathsf{C})$
- $\mathrm{Hom}_{\mathsf{C}/\sim}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y) \big/ \sim$.

There is a natural quotient functor $\pi : \mathsf{C} \to \mathsf{C}/\sim$ which has the universal property that if $F : \mathsf{C} \to \mathsf{D}$ is a functor which satisfies $g \sim h \Rightarrow F(g) = F(h)$ then F uniquely factors through π.

For any space X, we aim to get a congruence relation on $\mathcal{P}(X)$ given by path homotopy. Note that we can only define a path homotopy between two paths in the usual way if the paths have the same length. But we are allowing for paths of different length and such paths will remain unrelated if we use the usual path homotopy as our congruence relation. This is certainly undesirable as we would like all constant paths on a fixed point to be homotopic. So we will say that two paths are "related" if they become path-homotopic after appending constant paths to them to get the lengths equal. Here is the formal construction (I saw this approach in Ronald Brown's *Topology and Groupoids*):

For $x, y \in X$ let $\mathcal{P}_l(x,y)$ be the set of paths between x and y of duration l (for instance $\mathcal{P}_0(x,y) = \emptyset$ if $x \neq y$ and $\mathcal{P}_0(x,x) = \{\mathbf{c}_x\}$). We say that $\gamma, \delta \in \mathcal{P}_l(x,y)$ are *path-homotopic* and write $\gamma \approx \delta$ if there is a continuous map

$$H : [0,1] \times [0,l] \to X$$

such that

- $H(0,t) = \gamma(t)$,
- $H(1,t) = \delta(t)$,
- $H(s,0) = x$ for all s,
- $H(s,l) = y$ for all s.

In this case H is called a *path-homotopy* from γ to δ.

Proposition 2.1. *$\approx$ is an equivalence relation on $\mathcal{P}_l(x,y)$.*

Proof. Let $\gamma, \delta, \rho \in \mathcal{P}_l(x,y)$. The map

$$\begin{aligned} [0,1] \times [0,l] &\to X \\ (s,t) &\mapsto \gamma(t) \end{aligned}$$

defines a path-homotopy from γ to itself, hence $\approx$ is reflexive.

Suppose $\gamma \approx \delta$ via H. Define

$$\begin{aligned} G : [0,1] \times [0,l] &\to X \\ (s,t) &\mapsto H(1-s,t)\,. \end{aligned}$$

Observe that G is continuous and

- $G(0,t) = H(1,t) = \delta(t)$,
- $G(1,t) = H(0,t) = \gamma(t)$,
- $G(s,0) = H(1-s,0) = x$ for all s,
- $G(s,l) = H(1-s,l) = y$ for all s.

Thus $\delta \approx \gamma$ via G. So $\approx$ is symmetric.

Suppose $\gamma \approx \delta$ and $\delta \approx \rho$ via H and H', respectively. Define

$$\begin{aligned} G : [0,1] \times [0,l] &\to X \\ (s,t) &\mapsto \begin{cases} H(2s,t) & \text{if } 0 \le s \le 1/2, \\ H'(2s-1,t) & \text{if } 1/2 \le s \le 1. \end{cases} \end{aligned}$$

Observe that when $s = 1/2$ both cases in the definition gives $\delta(t)$ so G is continuous by the pasting lemma. And we have

- $G(0,t) = H(0,t) = \gamma(t)$,
- $G(1,t) = H'(1,t) = \rho(t)$,
- $G(s,0) = x$ for all s,
- $G(s,l) = y$ for all s.

Thus $\gamma \approx \rho$ via G. So $\approx$ is transitive. □

If we denote the set of *all* paths between x and y by $\mathcal{P}(x,y)$ then we have

$$\mathcal{P}(x,y) = \coprod_{l \ge 0} \mathcal{P}_l(x,y)$$

so we can regard $\approx$ as an equivalence relation on $\mathcal{P}(x,y)$ for every x,y (no interaction between paths of different lengths yet!).

Proposition 2.2. *$\approx$ is a congruence relation on $\mathcal{P}(X)$.*

Proof. Suppose $\gamma_1, \gamma_2 \in \mathcal{P}(x,y)$ and $\delta \in \mathcal{P}(y,z)$ with $\gamma_1 \approx \gamma_2$ (so $|\gamma_1| = |\gamma_2|$) via $H : [0,1] \times [0, |\gamma_i|] \to X$. Define

$$G : [0,1] \times [0, |\gamma_i| + |\delta|] \to X$$
$$(s,t) \mapsto \begin{cases} H(s,t) & \text{if } 0 \le t \le |\gamma_i|, \\ \delta(t - |\gamma_i|) & \text{if } |\gamma_i| \le t \le |\delta| + |\gamma_i|. \end{cases}$$

Since $H(s, |\gamma_i|) = y = \delta(0)$, by the pasting lemma G is continuous. And

- $G(0,t) = \begin{cases} \gamma_1(t) & \text{if } 0 \le t \le |\gamma_1|, \\ \delta(t - |\gamma_1|) & \text{if } |\gamma_1| \le t \le |\delta| + |\gamma_i|. \end{cases} = (\delta \cdot \gamma_1)(t),$
- $G(1,t) = \begin{cases} \gamma_2(t) & \text{if } 0 \le t \le |\gamma_2|, \\ \delta(t - |\gamma_2|) & \text{if } |\gamma_2| \le t \le |\delta| + |\gamma_2|. \end{cases} = (\delta \cdot \gamma_2)(t),$
- $G(s,0) = H(s,0) = x$ for all s,
- $G(s, |\gamma_i| + |\delta|) = \delta(|\delta|) = z$.

Thus $\delta \cdot \gamma_1 \approx \delta \cdot \gamma_2$ via G. In other words, path-homotopy is preserved under precomposing. A similar argument shows that it is preserved under postcomposing. □

As we indicated before $\approx$ is *not* the congruence relation we want to mod out from $\mathcal{P}(X)$. Let's introduce some notation first. For any non-negative real number d we have a constant path of duration d given by

$$[0,d] \to X$$
$$t \mapsto x$$

on each point x. We also write d for this path. The ambiguity about which point the constant path d stays at is resolved by the context. For instance if we write $d \cdot \gamma$ then d is understood to stay at $\mathbf{t}(\gamma)$ as that is the only way for the path-composite to make sense.

We define a new relation $\sim$ on $\mathcal{P}(x,y)$ by writing $\gamma \sim \delta$ if and only if there exist non-negative real numbers d, c such that $d \cdot \gamma \approx c \cdot \delta$. Note that this requires $d + |\gamma| = c + |\delta|$. Another remark is that $\mathbf{c}_x \sim d$ for any $d \in \mathbb{R}^{\ge 0}$, in other words every constant path on a point is related via $\sim$.

Proposition 2.3. *$\sim$ is an equivalence relation on $\mathcal{P}(x,y)$ which extends $\approx$.*

Proof. Let $\gamma, \delta, \rho \in \mathcal{P}(x,y)$. If $\gamma \approx \delta$, then by taking $d = c = 0$ we see that $\gamma \sim \delta$. So $\sim$ extends $\approx$. And $\sim$ is therefore reflexive.

Suppose $\gamma \sim \delta$. So there exists $d, c \in \mathbb{R}^{\ge 0}$ such that $d \cdot \gamma \approx c \cdot \delta$. Then $c \cdot \delta \approx d \cdot \gamma$ and hence $\delta \sim \gamma$. Therefore $\sim$ is symmetric.

Suppose $\gamma \sim \delta$ and $\delta \sim \rho$. So there are $d, c, b, a \in \mathbb{R}^{\ge 0}$ such that $d \cdot \gamma \approx c \cdot \delta$ and $b \cdot \delta \approx a \cdot \rho$. Then by using Proposition 2.2, we have

$$(b + d) \cdot \gamma = b \cdot d \cdot \gamma \approx b \cdot c \cdot \delta = c \cdot b \cdot \delta \approx c \cdot a \cdot \rho = (c + a) \cdot \rho\,.$$

Thus $\gamma \sim \rho$. Therefore $\sim$ is transitive. □

Before going on to show that $\sim$ is a congruence relation, let's digress and raise a legitimate concern about $\sim$: It extends $\approx$, but does it extend it too far? We would like $\sim$ to be a conservative extension in the sense that it does nothing more than incorporating paths of different lengths into path-homotopy. It would be undesirable for $\sim$ to relate two paths of same length which are not path-homotopic.

Proposition 2.4. *For every $l \in \mathbb{R}^{\geq 0}$, the relations $\sim$ and $\approx$ coincide on $\mathcal{P}_l(x, y)$.*

Proof. Let $\gamma, \delta \in \mathcal{P}_l(x, y)$ and $\gamma \sim \delta$. So there exists $d, c \in \mathbb{R}^{\geq 0}$ such that $d \cdot \gamma \approx c \cdot \delta$, say via H. Since $|\gamma| = |\delta| = l$, we have $c = d$. By definition, $H : [0, 1] \times [0, l + d] \to X$ is continuous and satisfies

- $H(0, t) = (d \cdot \gamma)(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq l\,, \\ y & \text{if } l \leq t \leq l + d\,. \end{cases}$
- $H(1, t) = (d \cdot \delta)(t) = \begin{cases} \delta(t) & \text{if } 0 \leq t \leq l\,, \\ y & \text{if } l \leq t \leq l + d\,. \end{cases}$
- $H(s, 0) = x$.
- $H(s, l + d) = y$.

Using H, we want to construct a continuous map $G : [0, 1] \times [0, l] \to X$ which is a path-homotopy from γ to δ. A reasonable attempt would be to seek a continuous map

$$f : [0, 1] \times [0, l] \to [0, 1] \times [0, l + d]$$

for which taking $G = H \circ f$ works. Observe that

- If $f(0, t) = (0, t)$ for every $t \in [0, l]$ then $G(0, t) = \gamma(t)$.
- If $f(1, t) = (1, t)$ for every $t \in [0, l]$ then $G(1, t) = \delta(t)$.
- If f sends $[0, 1] \times \{0\}$ into $[0, 1] \times \{0\}$ then $G(s, 0) = x$.
- If f sends $[0, 1] \times \{l\}$ into $[0, 1] \times \{l + d\} \cup \{0\} \times [l, l + d] \cup \{1\} \times [l, l + d]$ then $G(s, l) = y$.

So constructing such an f will suffice. At this point drawing the rectangles $[0, 1] \times [0, l]$ and $[0, 1] \times [0, l+d]$ and observing what the conditions on f mean geometrically is useful. Doing this, we see that all four of the conditions are only about the boundaries of the rectangles. Note that the boundary of $[0, 1] \times [0, l]$ is equal to the union $B \cup L \cup T \cup R$ (the letters stand for bottom, left, top, right respectively) where

$$\begin{aligned} B &= [0, 1] \times \{0\} \\ L &= \{0\} \times [0, l] \\ T &= [0, 1] \times \{l\} \\ R &= \{1\} \times [0, l]\,. \end{aligned}$$

Letting

$$\begin{aligned} T_1 &= \{0\} \times [l, l + d] \\ T_2 &= [0, 1] \times \{l + d\} \\ T_3 &= \{1\} \times [l, l + d], \end{aligned}$$

we see that the boundary of $[0, 1] \times [0, l + d]$ is equal to $B \cup L \cup T_1 \cup T_2 \cup T_3 \cup R$. So we can rewrite the conditions on f as

- f fixes L pointwise.
- f fixes R pointwise.

- $f(B) \subseteq B$.
- $f(T) \subseteq T_1 \cup T_2 \cup T_3$.

Define

$$f : [0,1] \times [0,l] \to [0,1] \times [0,l+d]$$
$$(s,t) \mapsto \frac{l-t}{l} \cdot (s,0) + \frac{t}{l} \cdot \begin{cases} (0, l+3sd) & \text{if } 0 \le s \le 1/3\,, \\ (3s-1, l+d) & \text{if } 1/3 \le s \le 2/3\,, \\ (1, l+3(1-s)d) & \text{if } 2/3 \le s \le 1\,. \end{cases}$$

The idea in the definition of f is to divide the domain into three rectangles and send the middle one to a trapezoid with the same bottom side but extended top side. I tried to write the definition so we see where each vertical line in the domain is sent to. For instance the line from $(0,1/3)$ to $(l,1/3)$ is sent to the line from $(1/3,0)$ to $(0,l+d)$.

By inspection we see that f is well-defined and continuous. And

- $f(0,t) = \frac{l-t}{l} \cdot (0,0) + \frac{t}{l} \cdot (0,l) = (0,t)$.
- $f(1,t) = \frac{l-t}{l} \cdot (1,0) + \frac{t}{l} \cdot (1,l) = (1,t)$.
- $f(s,0) = (s,0)$.
- $f(s,l) = \begin{cases} (0, l+3sd) & \text{if } 0 \le s \le 1/3\,, \\ (3s-1, l+d) & \text{if } 1/3 \le s \le 2/3\,, \\ (1, l+3(1-s)d) & \text{if } 2/3 \le s \le 1\,. \end{cases}$

  Here, the values in the first, second and third lines lie in T_1, T_2 and T_3, respectively.

□

Let's get back on track. We need the following lemma to show that $\sim$ is a congruence.

Lemma 2.5. *Given a path $\gamma \in \mathcal{P}(x,y)$ and $d \in \mathbb{R}^{\ge 0}$, we have $d \cdot \gamma \approx \gamma \cdot d$.*

Proof. Define

$$H : [0,1] \times [0, d+|\gamma|] \to X$$
$$(s,t) \mapsto \begin{cases} x & \text{if } 0 \le t \le sd, \\ \gamma(t-sd) & \text{if } sd \le t \le sd+|\gamma|, \\ y & \text{if } sd+|\gamma| \le t \le d+|\gamma|. \end{cases}$$

Note that H is well-defined and continuous, moreover

- $H(0,t) = (d \cdot \gamma)(t)$,
- $H(1,t) = (\gamma \cdot d)(t)$,
- $H(s,0) = x$ for all s,
- $H(s,1) = y$ for all s.

Thus $d \cdot \gamma \approx \gamma \cdot d$ via H. □

Proposition 2.6. *$\sim$ is a congruence relation on $\mathcal{P}(X)$.*

Proof. Let $\gamma_1, \gamma_2, \delta_1, \delta_2$ be paths in X such that $\gamma_1 \sim \gamma_2$, $\delta_1 \sim \delta_2$ and $\mathbf{t}(\gamma_i) = \mathbf{s}(\delta_i)$. So there exists $d_1, d_2, c_1, c_2 \in \mathbb{R}^{\geq 0}$ such that $d_1 \cdot \gamma_1 \approx d_2 \cdot \gamma_2$ and $c_1 \cdot \delta_1 \approx c_2 \cdot \delta_2$. Then using Proposition 2.2 and Lemma 2.5, we have

$$\begin{aligned}(c_1 + d_1) \cdot \delta_1 \cdot \gamma_1 &= d_1 \cdot c_1 \cdot \delta_1 \cdot \gamma_1 \\ &\approx c_1 \cdot \delta_1 \cdot d_1 \cdot \gamma_1 \\ &\approx c_2 \cdot \delta_2 \cdot d_2 \cdot \gamma_2 \\ &\approx d_2 \cdot c_2 \delta_2 \cdot \gamma_2 \\ = (c_2 + d_2) \cdot \delta_2 \cdot \gamma_2 \,.\end{aligned}$$

Hence $\delta_1 \cdot \gamma_1 \sim \delta_2 \cdot \gamma_2$. □

So we can form the quotient category $\mathcal{P}(X)/\sim$, which we denote by $\Pi(X)$. It has the universal property that every functor *from* $\mathcal{P}(X)$ which sends homotopic paths to the same morphism uniquely factors through $\Pi(X)$.

Proposition 2.7. $\Pi(X)$ *is a groupoid, that is, every morphism in* $\Pi(X)$ *is an isomorphism.*

Proof. Let $\gamma \in \mathcal{P}(x, y)$. We want to show that its equivalence class $[\gamma] \in \Pi(x, y)$ has an inverse. Let $l = |\gamma|$ and define

$$\begin{aligned}\delta : [0, l] &\to X \\ t &\mapsto \gamma(l - t) \,.\end{aligned}$$

Note that $\delta \in \mathcal{P}(y, x)$, hence $\delta \cdot \gamma \in \mathcal{P}(x, x)$ and

$$\begin{aligned}(\delta \cdot \gamma) : [0, 2l] &\to X \\ t &\mapsto \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq l, \\ \delta(t - l) = \gamma(2l - t) & \text{if } l \leq t \leq 2l. \end{cases}\end{aligned}$$

Now define

$$\begin{aligned}H : [0, 1] \times [0, 2l] &\to X \\ (s, t) &\mapsto \begin{cases} x & \text{if } 0 \leq t \leq sl, \\ \gamma(t - sl) & \text{if } sl \leq t \leq l, \\ \gamma((2 - s)l - t) & \text{if } l \leq t \leq (2 - s)l, \\ x & \text{if } (2 - s)l \leq t \leq 2l. \end{cases}\end{aligned}$$

Observe that H is well-defined and continuous. Moreover

- The map $H(0, -)$ is $\delta \cdot \gamma$.
- The map $H(1, -)$ is the constant path $2l$ at x.
- $H(s, 0) = x$ for all s.
- $H(s, 2l) = x$ for all s.

Thus $\delta \cdot \gamma \approx 2l$ via H. So $\delta \cdot \gamma \sim \mathbf{c}_x$, hence $[\delta] \cdot [\gamma] = [\mathbf{c}_x] = \mathrm{id}_x$ in $\Pi(X)$. A similar argument shows that $[\gamma] \cdot [\delta] = \mathrm{id}_y$. □

For every topological space X, we call $\Pi(X)$ the **fundamental groupoid** of X.

In conclusion, path-homotopy (taking into account paths of different duration) is a congruence relation on the path category for which the associated quotient category is the fundamental groupoid. It's important to observe that the "usual homotopy" is a congruence relation on Top. Recall that for continuous maps $f, g : X \to Y$ we write $f \simeq g$ if and only if there is a continuous map

$$H : [0,1] \times X \to Y$$

such that $H(0,-) = f$ and $H(1,-) = g$. In this situation we say that f is **homotopic** to g (or f can be *homotoped* to g) via H.

Proposition 2.8. *Homotopy is an equivalence relation on $C(X,Y)$ - the set of continuous maps from X to Y.*

Proof. Let $f, g, h \in C(X,Y)$. Via

$$\begin{aligned}[0,1] \times X &\to Y \\ (t,x) &\mapsto f(x)\end{aligned}$$

we have $f \simeq f$. So $\simeq$ is reflexive.

Suppose $f \simeq g$ via H. Define

$$\begin{aligned}G : [0,1] \times X &\to Y \\ (t,x) &\mapsto H(1-t, x)\,.\end{aligned}$$

Observe that G is continuous and $G(0,-) = H(1,-) = g$ and $G(1,-) = H(0,-) = f$. Thus $g \simeq f$. So $\simeq$ is symmetric.

Suppose $f \simeq g$ via H and $g \simeq h$ via H'. Define

$$\begin{aligned}G : [0,1] \times X &\to Y \\ (t,x) &\mapsto \begin{cases} H(2t, x) & \text{if } 0 \le t \le 1/2, \\ H'(2t-1, x) & \text{if } 1/2 \le t \le 1. \end{cases}\end{aligned}$$

Then since $H(1,-) = H'(0,-) = g$, the map G is well-defined and continuous. Furthermore $G(0,-) = H(0,-) = f$ and $G(1,-) = H'(1,-) = h$, thus $f \simeq h$ via G. So $\simeq$ is transitive. □

Proposition 2.9. *Homotopy is a congruence relation on* Top*.*

Proof. Let $f_1, f_2 \in C(X,Y)$ and $g \in C(Y,Z)$ with $f_1 \simeq f_2$ via $H : [0,1] \times X \to Y$. Then clearly $g \circ f_1 \simeq g \circ f_2$ via $g \circ H$. So homotopy is preserved under precomposing and by a similar argument it is preserved under postcomposing. □

So we can mod-out homotopies in Top. We call this quotient category hTop. It has the universal property that every functor *from* Top which sends homotopic maps to the same morphism uniquely factors through hTop. We use the notation $[X,Y]$ to denote the Hom-sets in hTop. In other words, $[X,Y]$ denotes the homotopy classes of maps from X to Y.

3. Π as a Functor

It is natural to suspect a functoriality since every space has its own fundamental groupoid. We write Grpd for the category of small groupoids, which is a full-subcategory of Cat.

Let $f : X \to Y$ be a continuous map. Applying the "path functor" to f, we get a functor $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ between the path categories. We want to show that f_* descends to a functor between the fundamental groupoids. By the universal property of the fundamental groupoids, it suffices to show that f_* respects $\sim$.

To that end, suppose $\gamma \sim \delta$ are paths in X. So there exist $d, c \in \mathbb{R}^{\geq 0}$ such that $d \cdot \gamma \approx c \cdot \delta$, say via H. Then clearly, $f \circ (d \cdot \gamma) \approx f \circ (c \cdot \delta)$ via $f \circ H$. We have

$$f \circ (d \cdot \gamma) = f_*(d \cdot \gamma) = f_*(d) \cdot f_*(\gamma) = d \cdot f_*(\gamma)$$

and similarly $f \circ (c \cdot \delta) = c \cdot f_*(\delta)$. Thus $f_*(\gamma) \sim f_*(\delta)$.

So we get a functor $\Pi(f) : \Pi(X) \to \Pi(Y)$ for every f. By using the fact that $\mathcal{P}$ is a functor and the uniqueness of the factorization through Π's it's straightforward to see that Π satisfies the functor axioms. We get the fundamental groupoid functor

$$\Pi : \mathsf{Top} \to \mathsf{Grpd}$$

It turns out that Π is a useful invariant. One reason is that it behaves well with homotopies.

Proposition 3.1. *If $f, g \in C(X, Y)$ are homotopic maps, then the functors $\Pi(f)$ and $\Pi(g)$ are naturally isomorphic.*

Proof. Let $H : [0,1] \times X \to Y$ be a homotopy from f to g. Then for every $x \in X$, we have a path

$$\begin{aligned} \gamma_x : [0,1] &\to Y \\ t &\mapsto H(t, x) \end{aligned}$$

in Y with $\mathbf{s}(\gamma_x) = H(0, x) = f(x)$ and $\mathbf{t}(\gamma_x) = H(1, x) = g(x)$, so $\gamma_x \in \mathcal{P}(f(x), g(x))$. Write $[\gamma_x]$ for the path-homotopy class of γ_x. Then $[\gamma_x]$ is a morphism in $\Pi(Y)$ from $f(x)$ to $g(x)$, actually an isomorphism since $\Pi(Y)$ is a groupoid.

We claim that the collection of $[\gamma_x]$'s define a natural transformation, hence a natural isomorphism, from $\Pi(f)$ to $\Pi(g)$. We need to show that for a path $\delta \in \mathcal{P}(x, x')$ in X, the diagram

$$\begin{CD} f(x) @>{\Pi(f)(\delta)=[f\circ\delta]}>> f(x') \\ @V{[\gamma_x]}VV @VV{[\gamma_{x'}]}V \\ g(x) @>{\Pi(g)(\delta)=[g\circ\delta]}>> g(x')\,. \end{CD}$$

commutes in $\Pi(Y)$. So it suffices to show that the paths $(g \circ \delta) \cdot \gamma_x$ and $\gamma_{x'} \cdot (f \circ \delta)$ are homotopic. Write $l = |\gamma|$. Note that

$$\begin{aligned} (g \circ \delta) \cdot \gamma_x : [0, 1 + l] &\to Y \\ t &\mapsto \begin{cases} H(t, x) & \text{if } t \in [0,1]\,, \\ g(\delta(t-1)) & \text{if } t \in [1, 1+l] \end{cases} \\ t &\mapsto H\left(\begin{cases} (t, x) & \text{if } t \in [0,1]\,, \\ (1, \delta(t-1)) & \text{if } t \in [1, 1+l] \end{cases} \right) \end{aligned}$$

and

$$\begin{aligned}\gamma_{x'} \cdot (f \circ \delta) : [0, l+1] &\to Y \\ t &\mapsto \begin{cases} f(\delta(t)) & \text{if } t \in [0, l]\,, \\ H(t-l, x') & \text{if } t \in [l, l+1]\,. \end{cases} \\ t &\mapsto H\left(\begin{cases} (0, \delta(t)) & \text{if } t \in [0, l]\,, \\ (t-l, x') & \text{if } t \in [l, l+1] \end{cases} \right).\end{aligned}$$

So it suffices to show that the paths

$$\begin{aligned}[0, l+1] &\to [0,1] \times X \\ t &\mapsto \begin{cases} (t, x) & \text{if } t \in [0,1]\,, \\ (1, \delta(t-1)) & \text{if } t \in [1, l+1] \end{cases}\end{aligned}$$

and

$$\begin{aligned}[0, l+1] &\to [0,1] \times X \\ t &\mapsto \begin{cases} (0, \delta(t)) & \text{if } t \in [0, l]\,, \\ (t-l, x') & \text{if } t \in [l, l+1] \end{cases}\end{aligned}$$

are homotopic (because precomposing such a path-homotopy with H gives the desired path-homotopy).

Let's introduce some notation to deal with this. In general if μ is a path in Y and ν is a path in Z such that $|\mu| = |\nu|$, we can define a path $\mu \times \nu$ in $Y \times Z$ in an obvious coordinate-wise fashion.

With this notation, letting $\iota : [0,1] \to [0,1]$ be the path given by the identity, the paths we are comparing above are $(l \cdot \iota) \times (\delta \cdot 1)$ and $(\iota \cdot l) \times (1 \cdot \delta)$. And indeed, by Lemma 2.5 and Lemma 3.2 these paths are homotopic. □

Lemma 3.2. *Let μ_1, μ_2 be paths in Y and ν_1, ν_2 be paths in Z. If $\mu_1 \approx \mu_2$, $\nu_1 \approx \nu_2$ and $|\mu_i| = |\nu_i|$, then $\mu_1 \times \nu_1 \approx \mu_2 \times \nu_2$.*

Proof. Write $|\mu_i| = |\nu_i| = l$. Say $\mu_1 \approx \mu_2$ via $H : [0,1] \times [0,l] \to Y$ and $\nu_1 \approx \nu_2$ via $K : [0,1] \times [0,l] \to Z$. Then it is straightforward to check that $\mu_1 \times \nu_1 \approx \mu_2 \times \nu_2$ via

$$\begin{aligned}H \times K : [0,1] \times [0,l] &\to Y \times Z \\ (s,t) &\mapsto (H(s,t), K(s,t))\,.\end{aligned}$$

□

As an immediate consequence of Proposition 3.1, we get the homotopy invariance of the fundamental groupoid.

Corollary 3.3. *If X and Y are homotopy equivalent spaces, then the groupoids $\Pi(X)$ and $\Pi(Y)$ are equivalent as categories.*

Remark 3.4. Due to my lack of knowledge of higher category theory, I am not making Π as categorical as it should be. In fact both Top and Grpd are 2-categories where the homotopies and natural transformations are 2-morphisms, respectively. And Π is a 2-functor. But I have never studied anything about 2-categories and probably already made a mistake in the last sentence, so I won't pursue this point of view here.

Seifert van-Kampen for Groupoids

In this section we state a groupoid version of the well-known Seifert van-Kampen theorem. Vaguely, it states that the functor $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ preserves *some* push-outs. Here is a precise statement.

Theorem 3.5. *Let $\mathcal{U}$ be an open covering of a space X such that every $U \in \mathcal{U}$ is path connected and $\mathcal{U}$ is closed under finite intersections. Consider the functor $\Pi|_{\mathcal{U}} : \mathcal{U} \to$ Grpd, regarding $\mathcal{U}$ as a subcategory of Top with morphisms as inclusions. Then Π sends the universal cone $\mathcal{U} \to X$ in Top to a universal cone in Grpd; hence $\Pi(X) \cong \operatorname{colim} \Pi|_{\mathcal{U}}$.*

Proof. Let $\mathcal{G}$ be a groupoid and $c : \Pi \to \mathcal{G}$ be a cone in Grpd. We want to construct a functor $F : \Pi(X) \to \mathcal{G}$ such that for every $U \in \mathcal{U}$ we have $F \circ \Pi(i_U) = c_U : \Pi(U) \to \mathcal{G}$, where $i_U : U \to X$ is the inclusion.

So start with a path $\gamma : x \to y$ in X, which is a continuous map $\gamma : [0, l] \to X$ with $\gamma(0) = x$ and $\gamma(l) = y$. Since $\{\gamma^{-1}(U) : U \in \mathcal{U}\}$ is an open covering of the compact metric space $[0, l]$, it has a positive Lebesgue number $\delta > 0$. So by dividing $[0, l]$ into finitely many subintervals of length less than δ, we obtain paths $\gamma_1, \dots, \gamma_n \in \mathcal{P}(X)$ which compose to γ such that each γ_k is contained in a single $U_k \in \mathcal{U}$. So for each k we may consider γ_k as a path in U_k, that is, a morphism in $\mathcal{P}(U_k)$. Then we have

$$\gamma = \mathcal{P}(i_{U_n})(\gamma_n) \cdots \mathcal{P}(i_{U_1})(\gamma_1) .$$

Modding out by path homotopy, the above equality becomes

$$[\gamma] = \Pi(i_{U_n})([\gamma_n]) \cdots \Pi(i_{U_1})([\gamma_1]) .$$

So if such a functor F exists, we necessarily have

$$(\star) \qquad F([\gamma]) = c_{U_n}([\gamma_n]) \cdots c_{U_1}([\gamma_1]) .$$

This establishes the uniqueness of such an F. To show existence, we are forced to define F as in $(\star)$. We need to show well-definition, so first suppose that we have a different subdivision of γ. That is, say we have another positive integer m, open sets $V_1, \dots, V_m \in \mathcal{U}$, and a path δ_r lying in V_r for each $r \in \{1, \dots, m\}$ such that

$$\gamma = \mathcal{P}(i_{V_m})(\delta_m) \cdots \mathcal{P}(i_{V_1})(\delta_1) .$$

TO BE CONTINUED. □

Covering Groupoids

It is rather well known that covering spaces satisfy the unique path lifting property. We abstract this notion in terms of groupoids.

Definition 3.6. Let $p : \mathcal{E} \to \mathcal{G}$ be a map of (small) groupoids. p is called a **covering map** if for every object e in $\mathcal{E}$, the functor $e/\mathcal{E} \to p(e)/\mathcal{E}$ of under-categories is bijective on objects.

That is, given objects $g \in \mathcal{G}$ and $e \in \mathcal{E}$ such that $p(e) = g$, every morphism out of g uniquely lifts to a morphism out of e. This uniqueness allows us to define a set-valued functor by taking inverse images.

Proposition 3.7. *Given a covering $p : \mathcal{E} \to \mathcal{G}$ of groupoids, the assignment*

$$g \mapsto p^{-1}(g) = \{e \in \mathrm{Obj}(E) : p(e) = g\}$$

defines a functor $F(p) : \mathcal{G} \to \mathsf{Set}$.

Proof. The action of $F(p)$ on objects is already specified. Let $\alpha : g \to g'$ be a morphism in $\mathcal{G}$. Now given $e \in p^{-1}(g)$, by the definition of covering there is a unique morphism $\lambda : e \to e'$ in $\mathcal{E}$ such that $p(\lambda) = \alpha$. So $p(e') = g'$, that is, $e' \in p^{-1}(g')$. We define $F(p)(\alpha)$ to be this recipe $e \mapsto e'$.

For the identity morphism $\mathrm{id}_g : g \to g$ in $\mathcal{G}$, given $e \in p^{-1}(g)$ the identity $\mathrm{id}_e : e \to e$ in $\mathcal{E}$ is necessarily the unique lift of id_g out of e. Hence $F(p)(\mathrm{id}_g)(e) = e$ for every $e \in p^{-1}(g)$, that is, $F(p)(\mathrm{id}_g) = \mathrm{id}_{p^{-1}(g)}$.

Given morphisms $\alpha : g \to g'$ and $\beta : g' \to g''$, fix $e \in p^{-1}(g)$. Let $\lambda : e \to e'$ be the unique lift of α out of e, and $\mu : e' \to e''$ be the unique lift of β out of e'. Then $\mu \circ \lambda : e \to e''$ is the unique lift of $\beta \circ \alpha$; thus

$$F(p)(\beta \circ \alpha)(e) = e'' = F(p)(\beta)(e') = F(p)(\beta)(F(p)(\alpha)(e)) = (F(p)(\beta) \circ F(p)(\alpha))(e)\,.$$

Since this holds for every $e \in p^{-1}(g)$, we get $F(p)(\beta \circ \alpha) = F(p)(\beta) \circ F(p)(\alpha)$. □

Next step is to show that the assignment $p \mapsto F(p)$ defines a functor F. The target category of F will naturally be the functor category $\mathsf{Set}^{\mathcal{G}}$. For the source of F we need a category $\mathsf{Cov}\text{-}\mathcal{G}$ of coverings. We define it in the obvious way, where the objects are covering maps to $\mathcal{G}$ and the morphisms are commutative triangles.

Proposition 3.8. *With the above specifications, $F : \mathsf{Cov}\text{-}\mathcal{G} \to \mathsf{Set}^{\mathcal{G}}$ defines a functor.*

Proof. We have already specified the action of F on the objects. For morphisms, let $q \in \mathsf{Cov}\text{-}G(\mathcal{E}, \mathcal{E}')$. So $p : \mathcal{E} \to \mathcal{G}$ and $p' : \mathcal{E}' \to \mathcal{G}$ are two coverings and $q : \mathcal{E} \to \mathcal{E}'$ is a functor such that $p' \circ q = p$.

We need to define a natural transformation $F(q) : F(p) \to F(p')$. So fix an object $g \in \mathcal{G}$. Define

$$F(q)_g : F(p)(g) = p^{-1}(g) \to p'^{-1}(g) = F(p')(g)$$
$$e \mapsto q(e)\,.$$

Here $F(q)_g$ is well-defined because if $e \in p^{-1}(g)$, then $p'(q(e)) = p(e) = g$, so $q(e) \in p'^{-1}(g)$. To check naturality, let $\alpha : g \to g'$ be a morphism in $\mathcal{G}$. The diagram

$$\begin{array}{ccc} F(p)(g) & \xrightarrow{F(p)(\alpha)} & F(p)(g') \\ {\scriptstyle F(q)_g}\downarrow & & \downarrow{\scriptstyle F(q)_{g'}} \\ F(p')(g) & \xrightarrow[F(p')(\alpha)]{} & F(p')(g') \end{array}$$

commutes: Let $e \in F(p)(g)$. Write $\lambda : e \to e'$ for the unique p-lift of α out of e, so

$$\left(F(q)_{g'} \circ F(p)(\alpha)\right)(e) = F(q)_{g'}(e') = q(e')\,.$$

On the other hand, $F(q)_g(e) = q(e)$ and since $p'(q(\lambda)) = p(\lambda) = \alpha$, the unique p'-lift of α out of $q(e)$ is $q(\lambda) : q(e) \to q(e')$. Thus

$$\left(F(p')(\alpha) \circ F(q)_g\right)(e) = F(p')(\alpha)(q(e)) = q(e')\,.$$

So the collection $\{F(q)_g : g \in \mathcal{G}\}$ defines a natural transformation $F(q) : F(p) \to F(p')$. This finishes the definition of F on morphisms.The identity and composition axioms are straightforward to check. $\square$

N